Complex Number 1

1. Given that $\sqrt{z} = \frac{2}{1-i} + 1 - 2i$, express the complex number z in the form $x + yi$.

$$\sqrt{z} = \frac{2}{1-i} + 1 - 2i = \frac{2(1+i)}{(1-i)(1+i)} + 1 - 2i = \frac{2(1+i)}{2} + 1 - 2i = 2 - i$$

$$z = (2-i)^2 = 3 - 4i$$

2. Calculate, in the form a + ib, where $a, b \in R$, the square root of $16 - 30i$.

Method 1

$$(a + ib)^2 = 16 - 30i$$

$$(a^2 - b^2) + 2abi = 16 - 30i$$

$$\begin{cases} a^2 - b^2 = 16 \ldots (1) \\ 2ab = -30 \ldots (2) \end{cases}$$

From (2), $b = -\frac{15}{a} \ldots (3)$

Substitute (3) in (1), $a^2 - \left(-\frac{15}{a}\right)^2 = 16 \Rightarrow a^4 - 16a^2 - 225 = 0 \Rightarrow (a^2+9)(a^2-25) = 0$

Since $a, b \in R$, $a^2 + 9 = 0$ has no solution, $a^2 = 25 \Rightarrow a = \pm 5$

From (3), $a + b\,i = 5 - 3i$ *or* $-5 + 3i$

Method 2

$$\sqrt{16-30i} = \sqrt{16 - 2(5)(3)i} = \sqrt{(5^2 - 3^2) - 2(5)(3)i} = \sqrt{(5^2 + (3i)^2) - 2(5)(3)i}$$

$$= \sqrt{5^2 - 2(5)(3)i + (3i)^2} = \sqrt{[\pm(5-3i)]^2} = \pm(5-3i)$$

3. Express the complex number $z = \sqrt{3} - i$ in its polar form. Hene, find $z^6 + \frac{1}{z^6}$ and $z^6 - \frac{1}{z^6}$.

$$|z| = \sqrt{\sqrt{3}^2 + (-1)^2} = 2 \ , \quad \arg z = tan^{-1}\frac{(-1)}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\therefore z = 2\left[cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

$z^6 = 2^6\left[cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]^6 = 2^6\left[\cos 6\left(-\frac{\pi}{6}\right) + i\sin\, 6\left(-\frac{\pi}{6}\right)\right]$, by de Moivre's Theorem

$$= 64[\cos(-\pi) + i\sin\,(-\pi)]$$

$$= -64$$

$$z^6 + \frac{1}{z^6} = -64 - \frac{1}{64} = -\mathbf{\frac{4097}{64}}, \quad z^6 - \frac{1}{z^6} = -64 + \frac{1}{64} = -\mathbf{\frac{4095}{64}}$$

4. If $z = 1 + 2i$ is a root of the equation $z^4 - z^3 + 4z^2 + 3z + 5 = 0$, express $z^4 - z^3 + 4z^2 + 3z + 5$ as a product of two quadratic factors. Hence, find the complex roots of the equation $z^4 - z^3 + 4z^2 + 3z + 5 = 0$.

If $z = 1 + 2i$ is a root, then $z = 1 - 2i$ is also a root.
Then $[z - (1 + 2i)][z - (1 - 2i)] = (z - 1)^2 - (2i)^2 = z^2 - 2z + 5$
is a factor of $z^4 - z^3 + 4z^2 + 3z + 5$.

By division, $\frac{z^4 - z^3 + 4z^2 + 3z + 5}{z^2 - 2z + 5} = z^2 + z + 1$.

Hence $z^4 - z^3 + 4z^2 + 3z + 5 = (z^2 - 2z + 5)(z^2 + z + 1)$

The given equation then becomes: $(z^2 - 2z + 5)(z^2 + z + 1) = 0$

$$z^2 - 2z + 5 = 0 \ \ or \ \ z^2 + z + 1 = 0$$

$$z = 1 \pm 2i \ or \ z = \frac{-1 \pm \sqrt{3}\, i}{2} \ .$$

5. Solve the equation $z^5 + 32 = 0$.

$$z^5 + 32 = 0 \Longrightarrow z^5 = -32 = 32(\cos\pi + i\,sin\,\pi) \Longrightarrow z = 2[\cos\pi + i\,sin\,\pi]^{\frac{1}{5}}$$

By de Moivre's Theorem, we have:

$$z = 2[\cos(2k\pi + \pi) + i\,sin\,(2k\pi + \pi)]^{\frac{1}{5}} = 2\left[\cos\left(\frac{2k\pi + \pi}{5}\right) + i\,sin\,\left(\frac{2k\pi + \pi}{5}\right)\right]$$, where k = 0,1,2,3,4.

$$z_0 = 2\left[\cos\left(\frac{\pi}{5}\right) + i\,sin\,\left(\frac{\pi}{5}\right)\right] \approx 1.6180339887499 + 1.1755705045849i$$

$$z_1 = 2\left[\cos\left(\frac{3\pi}{5}\right) + i\,sin\,\left(\frac{3\pi}{5}\right)\right] \approx -0.6180339887499 + 1.9021130325903i$$

$$z_2 = 2\left[\cos\left(\frac{5\pi}{5}\right) + i\,sin\,\left(\frac{5\pi}{5}\right)\right] = -2$$

$$z_4 = 2\left[\cos\left(\frac{7\pi}{5}\right) + i\,sin\,\left(\frac{7\pi}{5}\right)\right] \approx -0.6180339887499 - 1.9021130325903i$$

$$z_5 = 2\left[\cos\left(\frac{7\pi}{5}\right) + i\,sin\,\left(\frac{7\pi}{5}\right)\right] \approx 1.6180339887499 - 1.1755705045849i$$

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